ficients $\gamma_L^\mu \langle r_\mu | R_l^\mu \rangle$, the determinant of the coefficients should vanish:

$$\det\left|\left(F_{LL'}^{\mu\nu} - \frac{-i}{2AK_\mp R_\mp} D_L^\mu E_{L'}^\nu\right)\frac{k^{-1}\tan\eta_{l'}^\nu}{\langle r_\nu^c | j_{l'} k\rangle - \langle r_\nu^c | n_{l'} k\rangle \tan\eta_{l'}^\nu}\right| = 0 . \qquad \text{(A10)}$$

After factoring out

$$k^{-1}\tan\eta_{l'}^\nu / (\langle r_\nu^c | j_{l'} k\rangle - \langle r_\nu^c | n_{l'} k\rangle \tan\eta_{l'}^\nu) ,$$

Eq. (A10) becomes

$$\det\left| F_{LL'}^{\mu\nu} - \frac{-i}{2AK_\mp R_\mp} D_L^\mu E_{L'}^\nu \right| = 0 , \qquad \text{(A11)}$$

which is identical to Eq. (5.11).

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PHYSICAL REVIEW B VOLUME 4, NUMBER 2 15 JULY 1971

Equivalence of van der Waals Forces between Solids and the Surface-Plasmon Interaction

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It is shown that the well-known Lifshitz formula for the retarded van der Waals attraction between two solid half-spaces can be obtained from the zero-point energy of the interacting surface plasmons.

INTRODUCTION

Recently it was shown[1] that a nonretarded van der Waals attraction between solid half-spaces (separated by a gap) can be obtained from the interaction of the surface excitations on either side of the gap. This concept follows the basic idea, due to Casimir,[2] that the interaction between two media may be found from the total zero-point energy of all modes. In the following, it is shown that the equivalence between the two types of interaction also holds for the retarded case, thus leading to a considerable simplification of the general concept of van der Waals forces. Moreover, this equivalence establishes a connection between the large number of results on surface plasmons[3-6] and the important technological field of attraction between solids.[7]

DISPERSION RELATIONS FOR SURFACE PLASMONS

We consider a system consisting of two half-spaces (denoted by 1 and 2) separated by a vacuum gap (denoted by 3). The surfaces are assumed to be at $x=0$ and $x=d$.

In order to obtain dispersion relations for the surface plasmons we have to solve Maxwell's equation in 1, 2, and 3 and match the solutions according to continuity conditions. In a homogeneous medium characterized by a complex dielectric constant ϵ, magnetic permeability $\mu=1$, and vanishing external currents and charge densities, Maxwell's equations lead to the following wave equations for $\vec{E}$ and $\vec{B}$:

$$-\epsilon\ddot{\vec{E}}/c^2+\Delta\vec{E}=0\ , \tag{1}$$

$$-\epsilon\ddot{\vec{B}}/c^2+\Delta\vec{B}=0\ . \tag{2}$$

In order to obtain a solution for surface plasmons we used the ansatz

$$\vec{E}=(f_x(x),\ f_y(x),\ f_z(x))\,e^{i(k_y y+k_z z-\omega t)}\ . \tag{3}$$

For each of the components we find

$$\frac{\partial^2 f(x)}{\partial x^2}-K^2 f(x)=0\ , \tag{4}$$

where

$$K^2=k^2-\epsilon\,\omega^2/C^2, \tag{5}$$

$$k^2=k_y^2+k_z^2\ . \tag{6}$$

In the intervals

$$-\infty\leqslant x\leqslant 0\,\big|\,0\leqslant x\leqslant d\,\big|\,d\leqslant x\leqslant\infty$$

Eq. (4) leads to the following ansatz for f:

$$Ae^{K_1 x}\,\big|\,Be^{-K_3 x}+Ce^{K_3 x}\,\big|\,De^{-K_2 x}\ , \tag{7}$$

where A, B, C, and D are constants (standing for A_x, A_y, etc.) which are determined from continuity conditions at the surfaces at $x=0$ and $x=d$.

The magnetic vector $\vec{B}$ can be obtained from the relation

$$\dot{\vec{B}}+\mathrm{rot}\vec{E}=0\ , \tag{8}$$

where

$$\vec{B}=(b_x(x),\ b_y(x),\ b_z(x))\,e^{i(k_y y+k_z z-\omega t)}\ , \tag{9}$$

with

$$\frac{\partial^2 b(x)}{\partial x^2}-K^2 b(x)=0\ . \tag{10}$$

Insertion of Eqs. (3) and (9) leads to

$$(b_x,\ b_y,\ b_z)$$
$$=\omega^{-1}\left(k_y f_z-k_z f_y,\ k_z f_x+i\frac{\partial f_z}{\partial x}-k_y f_x-i\frac{\partial f_y}{\partial x}\right)\ . \tag{11}$$

If we take $\vec{k}$ to be parallel to the y direction, the linearity of Maxwell's equations permits us to consider the following cases separately:

$$\text{(a)}\ f_y\neq 0,\quad f_z=0, \tag{12}$$

$$\text{(b)}\ f_y=0,\quad f_z\neq 0. \tag{13}$$

Case (a) corresponds to electric waves (TM) with nonzero surface charge density. In this case all the boundary conditions are satisfied if

$$\epsilon f_x\ ,\quad \frac{\partial f_x}{\partial x} \tag{14}$$

are continuous.

Case (b) corresponds to magnetic waves with zero surface charge density everywhere. The sources of these waves are the electric currents $\vec{J}$ satisfying the equation $\mathrm{div}\,\vec{J}=0$ everywhere. In this case all the boundary conditions are satisfied if

$$f_z\ ,\quad \frac{\partial f_z}{\partial x} \tag{15}$$

are continuous.

Applying (14) to (7) and solving the resulting determinant then yields the dispersion relation

$$g_a\equiv\frac{(K_1+K_3\epsilon_1)(K_2+K_3\epsilon_2)}{(K_1-K_3\epsilon_1)(K_2-K_3\,\epsilon_2)}\,e^{2K_3 d}-1=0\ , \tag{16}$$

which is a generalization of the dispersion relation for surface plasmons derived by Economou[8] for $\epsilon_1=\epsilon_2$. The dispersion relation corresponding to (15) can be obtained from (16) by setting ϵ_1 and ϵ_2 equal to 1, but keeping them in K_1 and K_2. We obtain

$$g_b\equiv\frac{(K_1+K_3)(K_2+K_3)}{(K_1-K_3)(K_2-K_3)}\,e^{2K_3 d}-1=0\ . \tag{17}$$

(In the nonretarded case, g_b is infinite.) Although these excitations are not related to surface-charge-density oscillations, we will, in a generalized sense, call them surface plasmons also.

VAN DER WAALS ATTRACTION

Since ϵ is a function of the frequency ω, the dispersion relations (16) and (17) lead to a certain number of branches of $\omega(k)$. Then the zero-point energy U (per unit area) of our system can be obtained by summing $\frac{1}{2}\hbar\omega(k)$ over all branches and over all values of k. Proceeding as in Ref. 1 this sum can be expressed by the general relation

$$U=\frac{1}{(2\pi)^2}\int_0^\infty 2\pi k\,dk\left[\frac{1}{2\pi i}\oint\frac{\hbar\omega}{2}\times\left(\frac{\partial g_a(\omega)}{\partial\omega}\frac{1}{g_a(\omega)}+\frac{\partial g_b(\omega)}{\partial\omega}\frac{1}{g_b(\omega)}\right)d\omega\right]\ , \tag{18}$$

where the path has to be taken around the positive ω axis.

Here we used the fact that the contributions from the poles of g_a and g_b are d independent and thus do not contribute to the force $F=-\,\partial U/\partial d$ to be derived.

To calculate the integrals we put

$$\omega=i\xi\ , \tag{19}$$

$$ck=\xi(p^2-1)^{1/2}\ , \tag{20}$$

which leads to

$$p\xi/c=(k^2-\omega^2/c^2)^{1/2}\ , \tag{21}$$

$$\frac{\partial(\omega,k)}{\partial(\xi,p)}=\frac{i\xi}{c}\frac{p}{(p^2-1)^{1/2}}\ , \tag{22}$$

$$p=(c^2k^2/\xi^2+1)^{1/2}\operatorname{sgn}\xi\ . \tag{23}$$

Insertion into Eq. (17) gives

$$U=\frac{i\hbar}{8\pi^2c^2}\int p\,dp\oint\xi^3\,d\xi\times\left(\frac{\partial g_a(p,\xi)}{\partial\omega}\frac{1}{g_a(p,\xi)}+\frac{\partial g_b(p,\xi)}{\partial\omega}\frac{1}{g_b(p,\xi)}\right). \tag{24}$$

The quantity U includes a divergent contribution owing to the fact that the $\omega(k)$ curves do not go to zero for $d\rightarrow\infty$ (infinite energy of isolated surface-plasmon systems). This contribution is canceled by differentiation of U with respect to d, which yields the attraction force $F=-\,\partial U/\partial d$ resulting from interaction of the surface plasmons.

The derivatives in F are of the form

$$\frac{\partial}{\partial d}\left(\frac{\partial g(p,\xi)}{\partial\omega}\frac{1}{g(p,\xi)}\right)\quad . \tag{25}$$

Using the relations

$$i\frac{\partial g(p,\xi)}{\partial\omega}=-\frac{p^2-1}{p\xi}\frac{\partial g}{\partial p}+\frac{\partial g}{\partial\xi}\ , \tag{26}$$

$$\frac{\partial g}{\partial d}=\frac{2p\xi}{c}(g+1)\ , \tag{27}$$

the differentiations in (25) can easily be carried out to give

$$\frac{2}{ic}\left(\frac{1}{pg}+\frac{p^2-1}{g^2}\frac{\partial g}{\partial p}-\frac{p\xi}{g^2}\frac{\partial g}{\partial\xi}\right)\ . \tag{28}$$

Insertion into the equation for F and partial integration then gives

$$F=-\frac{\hbar^2}{2\pi^2c^3}\int_1^\infty p^2\,dp\int_0^\infty\xi^3\,d\xi\times\left(\frac{1}{g_a(p,\xi)}+\frac{1}{g_b(p,\xi)}\right)\ . \tag{29}$$

The dispersion relations $g_{a,b}(p,\xi)$ are obtained by inserting Eqs. (19)–(22) into (16) and (17); we find

$$g_a\equiv\frac{s_1+p\epsilon_1}{s_1-p\epsilon_1}\frac{s_2+p\epsilon_2}{s_2-p\epsilon_2}e^{2p\xi d/c}-1, \tag{30}$$

$$g_b\equiv\frac{s_1+p}{s_1-p}\frac{s_2+p}{s_2-p}e^{2p\xi d/c}-1\ , \tag{31}$$

where

$$s_{1,2}=(p^2-1+\epsilon_{1,2})^{1/2}\ . \tag{32}$$

The boundaries of the ξ and p integration in (29) are obtained as follows[9]: Since $\xi=-i\omega$, the path around the positive real ω axis in (18) transforms into a path around the negative imaginary ξ axis. This path can be shifted to the real ξ axis. Further, because of (23), the p integration is limited to the intervals $-\infty\leqslant p\leqslant 1$ and $1\leqslant p\leqslant\infty$. We obtain

$$\int dp\oint d\xi=\int_{\infty}^{-\infty}d\xi\left[\theta(-\xi)\int_{-1}^{-\infty}dp+\theta(\xi)\int_1^\infty dp\right] \tag{33}$$

$$=\int_{-\infty}^{0}d\xi\int_{-\infty}^{-1}dp-\int_0^\infty d\xi\int_1^\infty dp\ . \tag{34}$$

If we apply these integrals to the function $p^2\xi^3/g(p,\xi)$, the first term of (34) can be transformed by $\xi\rightarrow-\xi$ and $p\rightarrow-p$ to give the second one (including the sign). This argument is based on the fact that $g(p,\xi)=g(-p,-\xi)$. [The p's occurring in (30) and (31) have a factor of sign ξ.]

The force given by (29) is identical to that derived by Lifshitz[9] for the retarded van der Waals attraction of two solid half-spaces characterized by the complex dielectric constants $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$. Damping does not occur since $\operatorname{Im}\epsilon(i\omega)\equiv 0$.

CONCLUSIONS

We have shown that the well-known Lifshitz formula for the van der Waals attraction of two solid half-spaces is identical to the force resulting from the retarded interaction of surface plasmons. This result apparently is due to the fact that electromagnetic bulk excitations do not extend beyond surfaces[8] and thus do not contribute to an attractive force. Since the physical properties of surface plasmons are well understood, this new concept provides a much simpler intuitive understanding of van der Waals forces than the methods used before.[9,10–13] In particular, in the nonretarded limit, van der Waals forces are simply the attraction forces resulting from the interaction of the intrinsic surface-charge-density oscillations.

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Localization in One-Dimensional Disordered Systems*

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The basic idea that the convergence of the renormalized perturbation expression for the self-energy Δ_0 at a given energy is equivalent to the localizability of the eigenstates, if any, at this energy is applied to one-dimensional random systems, namely, electrons in the tight-binding approximation and phonons. For nearest-neighbor interactions all eigenstates are localized. If second-nearest-neighbor interactions are present, the possibility of the existence of extended states remains; we have shown that existing theories are unable to give a definite answer to the problem in this case.

I. INTRODUCTION

The study of disordered one-dimensional (1-D) systems has been focused on (a) numerical calculations of the average density of states, (b) theoretical efforts to show whether or not spectral gaps remain when disorder is introduced, and (c) theoretical and numerical studies to reveal the nature of the eigenstates, i.e., if they are localized or extended.

For the electronic case the early work was devoted to problems (b)[1–4] and (a)[3–8] above, although problem (c) was examined briefly in the work of Landauer and Helland.[4] Mott and Twose[9] were the first to suggest that all the electronic eigenfunctions in 1-D disordered systems are localized. Borland[10,11] was the first to present a rather general proof of this statement. A critical discussion of Borland's work is given in a review article by Halperin.[12] More recent work is discussed in review articles by Mott,[13] Hori,[14] Economou *et al.*,[15] and in books by Lieb and Mattis[16] and by Hori.[17]

For the problem of lattice vibrations in disordered 1-D systems the emphasis was on the calculation of the average spectral density. Since the pioneering work of Dyson,[18] many efforts have been made on analytical calculations of the spectrum. Dean,[19] in his remarkable numerical work, showed that the spectrum of an isotopically disordered linear chain has much fine structure with many well-defined peaks and valleys. Dean explained the existence of the peaks as due to states strongly localized around small islands of light masses surrounded by heavy masses. Hori *et al.*[20,21] gave a theoretical basis to Dean's finding by showing that such a fine structure should be expected for a wide variety of disordered systems. Dean[22] presented a proof, similar to that given by Borland for the electronic case, that all phonons in a 1-D disordered system are localized. It